Kompleksni brojevi

Zadatak 0.1. *Odrediti modul i argument za kompleksne brojeve:*

$z_1 = 1 + i, \quad z_2 = 1 - i, \quad z_3 = -1 + i, \quad z_4 = -1 - i.$

Rešenje. Za kompleksan broj $z = x + iy$, koji je zadat u algebarskom obliku, modul $|z|$ se računa po formuli $|z| = \sqrt{z \cdot \overline{z}} = \sqrt{x^2 + y^2}$. Ako kompleksan broj z predstavimo tačkom $M = (x, y)$ u kompleksnoj ravni, onda je rastojanje ρ tačke M od koordinatnog početka O jednako upravo modulu od z, tj. $\rho = |z|$. Ugao φ između pozitivnog dela x-ose i pravca neneula vektora $\overrightarrow{OM}$ naziva se argument kompleksnog broja z. Da bismo ga odredili, najpre ćemo definisati ugao $\varphi_0 \in \left(0, \frac{\pi}{2}\right)$, tako da je $\varphi_0 = \text{arctg}\left|\frac{y}{x}\right|$. Tada ugao $\varphi \in (-\pi, \pi]$ određujemo na sledeći način:

1. za $x > 0$ i $y > 0$ je $\varphi = \varphi_0$;
2. za $x < 0$ i $y > 0$ je $\varphi = \pi - \varphi_0$;
3. za $x > 0$ i $y < 0$ je $\varphi = -\varphi_0$;
4. za $x < 0$ i $y < 0$ je $\varphi = -\pi + \varphi_0$;
5. za $x > 0$ i $y = 0$ je $\varphi = 0$;
6. za $x = 0$ i $y > 0$ je $\varphi = \frac{\pi}{2}$;
7. za $x < 0$ i $y = 0$ je $\varphi = \pi$;
8. za $x = 0$ i $y < 0$ je $\varphi = -\frac{\pi}{2}$.

Jednostavno se proverava da brojevi z_1, z_2, z_3 i z_4 imaju isti modul $\rho = \sqrt{2}$. Takođe, φ_0 je za sva četiri broja isti i iznosi $\frac{\pi}{4}$. Odavde dobijamo da je $\varphi_1 = \frac{\pi}{4}$, $\varphi_2 = -\frac{\pi}{4}$, $\varphi_3 = \frac{3\pi}{4}$ i $\varphi_4 = -\frac{3\pi}{4}$. □

Zadatak 0.2. *Dati su kompleksni brojevi:*

a) $z_1 = \frac{1}{2} + i\frac{\sqrt{3}}{2}$, b) $z_2 = -\sqrt{2} + i\sqrt{2}$,

c) $z_3 = -\sqrt{3} - i$, d) $z_4 = 32 - 32i$.

Odrediti njihov eksponencijalni i trigonometrijski oblik.

Rešenje. Eksponencijalni oblik kompleksnog broja dat je sa $z = \rho e^{i\varphi}$. Primenjujući Ojlerovu formulu $e^{i\varphi} = \cos\varphi + i\sin\varphi$, dobijamo trigonometrijski oblik $z = \rho(\cos\varphi + i\sin\varphi)$.

a) Za kompleksan broj $z_1 = \frac{1}{2} + i\frac{\sqrt{3}}{2}$, najpre je potrebno odrediti ρ i φ. Kako je $x = \frac{1}{2}$ i $y = \frac{\sqrt{3}}{2}$, to je

$$\rho = \sqrt{x^2 + y^2} = \sqrt{\left(\frac{1}{2}\right)^2 + \left(\frac{\sqrt{3}}{2}\right)^2} = 1,$$

$$\varphi_0 = \operatorname{arctg}\left|\frac{y}{x}\right| = \operatorname{arctg}\left|\frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}}\right| = \operatorname{arctg}\sqrt{3} = \frac{\pi}{3} \quad \text{i} \quad \varphi = \varphi_0 = \frac{\pi}{3},$$

pa je eksponencijalni oblik kompleksnog broja z_1 dat sa $z_1 = e^{i\frac{\pi}{3}}$, a njegov trigonometrijski oblik sa $z_1 = \cos\frac{\pi}{3} + i\sin\frac{\pi}{3}$.

b) Za $z_2 = -\sqrt{2} + i\sqrt{2}$ je $\rho = 2$, $\varphi_0 = \operatorname{arctg} \mid \frac{\sqrt{2}}{-\sqrt{2}} \mid = \operatorname{arctg} 1 = \frac{\pi}{4}$, $\varphi = \pi - \varphi_0 = \frac{3\pi}{4}$, pa je

$$z_2 = 2e^{i\frac{3\pi}{4}} = 2(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}).$$

c) Za $z_3 = -\sqrt{3} - i$ je $\rho = 2$, $\varphi_0 = \operatorname{arctg}\left|\frac{-1}{-\sqrt{3}}\right| = \operatorname{arctg}\frac{\sqrt{3}}{3} = \frac{\pi}{6}$, $\varphi = -\pi + \varphi_0 = -\frac{5\pi}{6}$, pa je

$$z_3 = 2e^{-i\frac{5\pi}{6}} = 2(\cos\frac{5\pi}{6} - i\sin\frac{5\pi}{6}).$$

d) Za $z_4 = 32 - 32i$ je $\rho = 32\sqrt{2}$, $\varphi_0 = \operatorname{arctg}\left|\frac{-32}{32}\right| = \operatorname{arctg} 1 = \frac{\pi}{4}$, $\varphi = -\varphi_0 = -\frac{\pi}{4}$, pa je

$$z_4 = 32\sqrt{2}e^{i\frac{\pi}{4}} = 32\sqrt{2}(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}).$$

□

Zadatak 0.3. *Dati su kompleksni brojevi:*

a) $e^{i2k\pi}$, $k \in \mathbb{Z}$; b) $e^{i(\frac{\pi}{2}+2k\pi)}$, $k \in \mathbb{Z}$;

c) $e^{i(2k+1)\pi}$, $k \in \mathbb{Z}$; d) $e^{i(-\frac{\pi}{2}+2k\pi)}$, $k \in \mathbb{Z}$.

Odrediti njihov algebarski oblik.

Rešenje. a) $e^{i2k\pi} = \cos(2k\pi) + i\sin(2k\pi) = 1.$

b) $e^{i(\frac{\pi}{2}+2k\pi)} = e^{i\frac{\pi}{2}} \cdot e^{i2k\pi} = e^{i\frac{\pi}{2}} \cdot 1 = \cos\frac{\pi}{2} + i\sin\frac{\pi}{2} = i.$

c) $e^{i(2k+1)\pi} = e^{i2k\pi} \cdot e^{i\pi} = 1 \cdot e^{i\pi} = \cos\pi + i\sin\pi = -1.$

d) $e^{i(-\frac{\pi}{2}+2k\pi)} = e^{-i\frac{\pi}{2}} \cdot e^{i2k\pi} = e^{-i\frac{\pi}{2}} \cdot 1 = \cos(-\frac{\pi}{2}) + i\sin(-\frac{\pi}{2})$

$= \cos\frac{\pi}{2} - i\sin\frac{\pi}{2} = -i.$

□

Zadatak 0.4. *Izračunati:*

a) $(1+i)^{10} + (1-i)^{10}$; b) $\dfrac{(\frac{1}{2}+i\frac{\sqrt{3}}{2})^7 - (\frac{1}{2}-i\frac{\sqrt{3}}{2})^7}{(\frac{1}{2}+i\frac{\sqrt{3}}{2})^7 \cdot (\frac{1}{2}-i\frac{\sqrt{3}}{2})^7}$.

Rešenje. a) Za stepenovanje kompleksnih brojeva pogodno je koristiti njihov eksponencijalni oblik. Kako je $1+i = \sqrt{2}e^{i\frac{\pi}{4}}$ i $1-i = \sqrt{2}e^{-i\frac{\pi}{4}}$, to je

$$(1+i)^{10} + (1-i)^{10} = \left(\sqrt{2}e^{i\frac{\pi}{4}}\right)^{10} + \left(\sqrt{2}e^{-i\frac{\pi}{4}}\right)^{10} = 2^5 e^{i\frac{5\pi}{2}} + 2^5 e^{-i\frac{5\pi}{2}}$$
$$= 32e^{2\pi i}e^{i\frac{\pi}{2}} + 32e^{-2\pi i}e^{-i\frac{\pi}{2}} = 32i - 32i = 0.$$

b) Kako je $\frac{1}{2} + i\frac{\sqrt{3}}{2} = e^{\frac{i\pi}{3}}$ i $\frac{1}{2} - i\frac{\sqrt{3}}{2} = e^{-\frac{i\pi}{3}}$, to je

$$\frac{(\frac{1}{2}+i\frac{\sqrt{3}}{2})^7 - (\frac{1}{2}-i\frac{\sqrt{3}}{2})^7}{(\frac{1}{2}+i\frac{\sqrt{3}}{2})^7 \cdot (\frac{1}{2}-i\frac{\sqrt{3}}{2})^7} = \frac{(e^{\frac{i\pi}{3}})^7 - (e^{-\frac{i\pi}{3}})^7}{(e^{\frac{i\pi}{3}})^7 \cdot (e^{-\frac{i\pi}{3}})^7} = \frac{e^{\frac{i7\pi}{3}} - e^{-\frac{i7\pi}{3}}}{e^{\frac{i7\pi}{3}} \cdot e^{-\frac{i7\pi}{3}}}$$
$$= e^{i2\pi} \cdot e^{\frac{i\pi}{3}} - e^{-i2\pi} \cdot e^{-\frac{i\pi}{3}} = e^{\frac{i\pi}{3}} - e^{-\frac{i\pi}{3}}$$
$$= (\cos\frac{\pi}{3} + i\sin\frac{\pi}{3}) - (\cos\frac{\pi}{3} - i\sin\frac{\pi}{3}) = 2i\sin\frac{\pi}{3} = i\sqrt{3}.$$

□

Zadatak 0.5. *Dokazati:*

a) $\sin\alpha = \frac{1}{2i}\left(e^{i\alpha} - e^{-i\alpha}\right);$

b) $\cos\alpha = \frac{1}{2}\left(e^{i\alpha} + e^{-i\alpha}\right);$

c) $e^{i\alpha} - 1 = 2i\; e^{i\frac{\alpha}{2}} \sin\frac{\alpha}{2}.$

Rešenje. Na osnovu Ojlerove teoreme imamo da je $e^{i\alpha} = \cos\alpha + i\sin\alpha$ i $e^{-i\alpha} = \cos\alpha - i\sin\alpha$. Oduzimanjem (redom) levih i desnih strana ovih jednakosti, dobija se tvrđenje a), dok se njihovim sabiranjem dobija tvrđenje b). Dokažimo da važi c). Na osnovu tvrđenja a) imamo da je

$$e^{i\alpha} - 1 = e^{i\frac{\alpha}{2}+i\frac{\alpha}{2}} - e^{i\frac{\alpha}{2}-i\frac{\alpha}{2}} = e^{i\frac{\alpha}{2}}\left(e^{i\frac{\alpha}{2}} - e^{-i\frac{\alpha}{2}}\right) = 2ie^{i\frac{\alpha}{2}}\sin\frac{\alpha}{2}.$$

□

Zadatak 0.6. *Izračunati:*

$$\sqrt[3]{1+i}\;.$$

Rešenje. Neka je $n \in \mathbb{N}$. Sva rešenja jednačine $z^n = \rho e^{i\varphi}$ su n-ti koreni iz kompleksnog broja $\rho e^{i\varphi}$ i pod $\sqrt[n]{\rho e^{i\varphi}}$ podrazumeva se skup vrednosti $\{z_0, z_1, \ldots, z_{n-1}\}$, gde je

$$z_k = \sqrt[n]{w} = \sqrt{\rho}\; e^{i\frac{\varphi+2k\pi}{n}}, \qquad k = 0, 1, 2, \ldots, n-1. \tag{1}$$

Kompleksan broj $z = 1 + i$ najpre treba predstaviti u eksponencijalnom obliku $z = \sqrt{2}e^{i\frac{\pi}{4}}$. Dalje je

$$z_k = \sqrt[6]{2}e^{i\frac{\frac{\pi}{4}+2k\pi}{3}}, \qquad k = 0, 1, 2,$$

odnosno

$$z_k = \sqrt[6]{2}e^{i\frac{\pi+8k\pi}{12}}, \qquad k = 0, 1, 2.$$

Dakle,

$$\begin{aligned}
z_0 &= \sqrt[6]{2}e^{i\frac{\pi}{12}} = \sqrt[6]{2}\left(\cos\frac{\pi}{12} + i\sin\frac{\pi}{12}\right),\\
z_1 &= \sqrt[6]{2}e^{i\frac{9\pi}{12}} = \sqrt[6]{2}e^{i\frac{3\pi}{4}} = \sqrt[6]{2}\left(\cos\frac{3\pi}{4} + i\sin\frac{3\pi}{4}\right)\\
&= \sqrt[6]{2}\left(-\frac{1}{\sqrt{2}} + i\frac{1}{\sqrt{2}}\right) = -\frac{1}{\sqrt[3]{2}} + i\frac{1}{\sqrt[3]{2}}\;,\\
z_2 &= \sqrt[6]{2}e^{i\frac{17\pi}{12}} = \sqrt[6]{2}e^{i\left(\frac{18\pi}{12}-\frac{\pi}{12}\right)} = \sqrt[6]{2}e^{i\frac{3\pi}{2}}\,e^{-i\frac{\pi}{12}} = -i\sqrt[6]{2}e^{-i\frac{\pi}{12}}\\
&= -i\sqrt[6]{2}\left(\cos\frac{\pi}{12} - i\sin\frac{\pi}{12}\right) = -\sqrt[6]{2}\left(\sin\frac{\pi}{12} + i\cos\frac{\pi}{12}\right).
\end{aligned}$$

Za predstavljnje brojeva z_0 i z_2 u algebarskom obliku potrebno je izračunati $\cos\frac{\pi}{12}$ i $\sin\frac{\pi}{12}$. Iskoristićemo formule

$$\sin\frac{x}{2}=\sqrt{\frac{1-\cos x}{2}} \quad \text{i} \quad \cos\frac{x}{2}=\sqrt{\frac{1+\cos x}{2}}.$$

Odatle je

$$\begin{aligned}\sin\frac{\pi}{12} &= \sqrt{\frac{1-\cos\frac{\pi}{6}}{2}}=\sqrt{\frac{1-\frac{\sqrt{3}}{2}}{2}}=\frac{1}{2}\sqrt{2-\sqrt{3}}\,,\\ \cos\frac{\pi}{12} &= \sqrt{\frac{1+\cos\frac{\pi}{6}}{2}}=\sqrt{\frac{1+\frac{\sqrt{3}}{2}}{2}}=\frac{1}{2}\sqrt{2+\sqrt{3}}\,.\end{aligned}$$

Dalje, na osnovu formula

$$\begin{aligned}\sqrt{a+\sqrt{b}} &= \sqrt{\frac{a+\sqrt{a^2-b}}{2}}+\sqrt{\frac{a-\sqrt{a^2-b}}{2}}\,,\\ \sqrt{a-\sqrt{b}} &= \sqrt{\frac{a+\sqrt{a^2-b}}{2}}-\sqrt{\frac{a-\sqrt{a^2-b}}{2}}\,,\end{aligned}$$

dobijamo da je

$$\begin{aligned}\sin\frac{\pi}{12} &= \frac{1}{2}\sqrt{2-\sqrt{3}}=\frac{1}{2}\left(\sqrt{\frac{3}{2}}-\sqrt{\frac{1}{2}}\right)=\frac{\sqrt{3}-1}{2\sqrt{2}}\,,\\ \cos\frac{\pi}{12} &= \frac{1}{2}\sqrt{2+\sqrt{3}}=\frac{1}{2}\left(\sqrt{\frac{3}{2}}+\sqrt{\frac{1}{2}}\right)=\frac{\sqrt{3}+1}{2\sqrt{2}}\,.\end{aligned}$$

Odavde konačno dobijamo

$$\begin{aligned}z_0 &= \sqrt[6]{2}\left(\frac{\sqrt{3}+1}{2\sqrt{2}}+i\frac{\sqrt{3}-1}{2\sqrt{2}}\right)=\frac{1}{\sqrt[3]{16}}\left(\left(\sqrt{3}+1\right)+i\left(\sqrt{3}-1\right)\right),\\ z_1 &= -\frac{1}{\sqrt[3]{2}}+i\frac{1}{\sqrt[3]{2}}\,,\\ z_2 &= \sqrt[6]{2}\left(\frac{\sqrt{3}-1}{2\sqrt{2}}-i\frac{\sqrt{3}+1}{2\sqrt{2}}\right)=\frac{1}{\sqrt[3]{16}}\left(\left(\sqrt{3}-1\right)+i\left(\sqrt{3}+1\right)\right).\end{aligned}$$

□

Zadatak 0.7. *Izračunati:*

$$\sqrt[4]{2 - i\sqrt{12}}\ .$$

Rešenje. Odredimo ρ i φ za kompleksan broj $2 - i\sqrt{12}$.

$$\begin{aligned}
\rho &= \sqrt{2^2 + \left(\sqrt{12}\right)^2} = 4 \\
\varphi_0 &= \operatorname{arctg}\left|\frac{-\sqrt{12}}{2}\right| = \operatorname{arctg}\sqrt{3} = \frac{\pi}{3} \implies \varphi = -\frac{\pi}{3}
\end{aligned}$$

Dakle, treba izračunati $\sqrt[4]{2 - i\sqrt{12}} = \sqrt[4]{4e^{-i\frac{\pi}{3}}}$. Prema (1), za $n = 4$, dobija se

$$\begin{aligned}
z_k &= \sqrt[4]{4}e^{i\frac{-\frac{\pi}{3}+2k\pi}{4}}, \qquad k = 0, 1, 2, 3, \\
z_k &= \sqrt{2}e^{i\frac{(6k-1)\pi}{12}}, \qquad k = 0, 1, 2, 3,
\end{aligned}$$

odnosno,

$$\begin{aligned}
z_0 &= \sqrt{2}e^{-i\frac{\pi}{12}} = \sqrt{2}\left(\cos\frac{\pi}{12} - i\sin\frac{\pi}{12}\right), \\
z_1 &= \sqrt{2}e^{i\frac{5\pi}{12}} = \sqrt{2}e^{i\left(\frac{\pi}{2} - \frac{\pi}{12}\right)} \\
&= \sqrt{2}e^{i\frac{\pi}{2}}e^{-i\frac{\pi}{12}} = \sqrt{2}i\left(\cos\frac{\pi}{12} - i\sin\frac{\pi}{12}\right) \\
&= \sqrt{2}\left(\sin\frac{\pi}{12} + i\cos\frac{\pi}{12}\right), \\
z_2 &= \sqrt{2}e^{i\frac{11\pi}{12}} = \sqrt{2}e^{i\left(\pi - \frac{\pi}{12}\right)} = \sqrt{2}e^{i\pi}e^{-i\frac{\pi}{12}} \\
&= -\sqrt{2}e^{-i\frac{\pi}{12}} = -z_0, \\
z_3 &= \sqrt{2}e^{i\frac{17\pi}{12}} = \sqrt{2}e^{i\left(\pi + \frac{5\pi}{12}\right)} = \sqrt{2}e^{i\pi}e^{i\frac{5\pi}{12}} \\
&= -\sqrt{2}e^{i\frac{5\pi}{12}} = -z_1.
\end{aligned}$$

Kako je iz prethodnog zadatka

$$\cos\frac{\pi}{12} = \frac{\sqrt{3}+1}{2\sqrt{2}} \quad \text{i} \quad \sin\frac{\pi}{12} = \frac{\sqrt{3}-1}{2\sqrt{2}},$$

to je

$$\begin{aligned} z_0 &= \frac{\sqrt{3}+1}{2} - i\frac{\sqrt{3}-1}{2\sqrt{2}}, \\ z_1 &= \frac{\sqrt{3}-1}{2} + i\frac{\sqrt{3}+1}{2\sqrt{2}}, \\ z_2 &= -\frac{\sqrt{3}+1}{2} + i\frac{\sqrt{3}-1}{2\sqrt{2}}, \\ z_3 &= -\frac{\sqrt{3}-1}{2} - i\frac{\sqrt{3}+1}{2\sqrt{2}}. \end{aligned}$$

□

Zadatak 0.8. *Rešiti jednačinu:*

$$\frac{iz^4-1}{iz^4+1} = -i.$$

Rešenja napisati u algebarskom obliku i predstaviti ih u kompleksnoj ravni.

Rešenje.

$$\frac{iz^4-1}{iz^4+1} = -i \iff iz^4 - 1 = z^4 - i \iff (1-i)z^4 = -1+i$$

$$\iff z^4 = \frac{-1+i}{1-i} \iff z^4 = -1 \iff z = \sqrt[4]{-1}$$

$$\iff z = \sqrt[4]{e^{i\pi}} \iff z_k = e^{i\frac{\pi+2k\pi}{4}}, \quad k = 0,1,2,3.$$

$$z_0 = e^{\frac{i\pi}{4}} = \cos\frac{\pi}{4} + i\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2},$$

$$z_1 = e^{\frac{i3\pi}{4}} = e^{i(\pi-\frac{\pi}{4})} = e^{i\pi}\cdot e^{-i\frac{\pi}{4}} = -e^{-i\frac{\pi}{4}} = -(\cos\frac{\pi}{4} - i\sin\frac{\pi}{4}) = -\frac{\sqrt{2}}{2} + i\frac{\sqrt{2}}{2},$$

$$z_2 = e^{\frac{i5\pi}{4}} = e^{i(\pi+\frac{\pi}{4})} = e^{i\pi}\cdot e^{i\frac{\pi}{4}} = -e^{i\frac{\pi}{4}} = -(\cos\frac{\pi}{4} + i\sin\frac{\pi}{4}) = -\frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2},$$

$$z_3 = e^{\frac{i7\pi}{4}} = e^{i(2\pi-\frac{\pi}{4})} = e^{i2\pi}\cdot e^{-i\frac{\pi}{4}} = e^{-i\frac{\pi}{4}} = \cos\frac{\pi}{4} - i\sin\frac{\pi}{4} = \frac{\sqrt{2}}{2} - i\frac{\sqrt{2}}{2}.$$

□

Zadatak 0.9. *Dokazati formulu:*

$$\left(\sqrt{a+bi}\right)_{1,2} = \pm\left(\sqrt{\frac{\sqrt{a^2+b^2}+a}{2}} + i\sqrt{\frac{\sqrt{a^2+b^2}-a}{2}}\right), \text{ ako je } b>0$$

$$\left(\sqrt{a+bi}\right)_{1,2} = \pm\left(\sqrt{\frac{\sqrt{a^2+b^2}+a}{2}} - i\sqrt{\frac{\sqrt{a^2+b^2}-a}{2}}\right), \text{ ako je } b<0.$$

Rešenje. Pretpostavimo da je $z = x + iy$, takav da je $z = \sqrt{a+bi}$. Tada je $(x+iy)^2 = a + bi$, odnosno $x^2 - y^2 + 2xyi = a + bi$, odakle sledi da je $x^2 - y^2 = a$ i $2xy = b$. Odavde je $y = \frac{b}{2x}$, pa dobijamo

$$x^2 + \frac{b^2}{4x^2} = a \iff \frac{4x^4 - 4ax^2 + b^2}{4x^2} = 0 \iff \left(x^2\right)_{1,2} = \frac{a \pm \sqrt{a^2+b^2}}{2}$$

Kako je $a - \sqrt{a^2+b^2} < 0$, to je prethodna jednačina ekvivalentna sa

$$x^2 = \frac{a+\sqrt{a^2+b^2}}{2},$$

odnosno, ekvivalentna je sa

$$x_{1,2} = \pm\sqrt{\frac{a+\sqrt{a^2+b^2}}{2}}.$$

Dalje, dobijamo da je

$$y_{1,2} = \pm\frac{b}{2}\sqrt{\frac{2}{a+\sqrt{a^2+b^2}}}\cdot\sqrt{\frac{\sqrt{a^2+b^2}-a}{\sqrt{a^2+b^2}-a}}$$

$$= \pm\frac{b}{2}\sqrt{\frac{2\left(\sqrt{a^2+b^2}-a\right)}{b^2}} = \pm\frac{b}{|b|}\sqrt{\frac{\sqrt{a^2+b^2}-a}{2}}.$$

Sada, ako je $b > 0$, dobijamo da je

$$y_{1,2} = \pm\sqrt{\frac{\sqrt{a^2+b^2}-a}{2}},$$

a ako je $b < 0$, onda je

$$y_{1,2} = \mp\sqrt{\frac{\sqrt{a^2+b^2}-a}{2}}.$$

□

Zadatak 0.10. *Rešiti jednačinu:*

$z^2 - (2+i)\,z + (-1+7i) = 0.$

Rešenje. Za datu kvadratnu jednačinu $z^2 - (2+i)\,z + (-1+7i) = 0$ je

$$D = (2+i)^2 - 4\,(-1+7i) = 7 - 24i,$$

pa je

$$\left(\sqrt{7-24i}\right)_{1,2} = \pm\left(\sqrt{\frac{\sqrt{7^2+24}+7}{2}} - i\sqrt{\frac{\sqrt{7^2+24}-7}{2}}\right) = \pm\,(4-3i)\,,$$

odakle dobijamo da je

$$z_{1,2} = \frac{(2+i)\pm(4-3i)}{2},$$

odnosno

$$z_1 = 3 - i \quad \text{i} \quad z_2 = -1 + 2i.$$

□

Zadatak 0.11. *Ako je* $z = \frac{1+i}{\sqrt{2}}$, *naći* $1 + z + z^2 + \cdots + z^{30}$.

Rešenje. Kako je $z = \frac{1+i}{\sqrt{2}} = e^{\frac{i\pi}{4}}$, to je

$$1 + z + z^2 + \cdots + z^{30} = \frac{1-z^{31}}{1-z} = \frac{1-(e^{\frac{i\pi}{4}})^{31}}{1-e^{\frac{i\pi}{4}}} = \frac{1-e^{\frac{i31\pi}{4}}}{1-e^{\frac{i\pi}{4}}} = \frac{1-e^{i8\pi}e^{-i\frac{\pi}{4}}}{1-e^{\frac{i\pi}{4}}}$$

$$= \frac{1-e^{-i\frac{\pi}{4}}}{1-e^{\frac{i\pi}{4}}}\cdot\frac{1-e^{-i\frac{\pi}{4}}}{1-e^{-\frac{i\pi}{4}}} = \frac{(1-e^{-i\frac{\pi}{4}})^2}{1-(e^{i\frac{\pi}{4}}+e^{-i\frac{\pi}{4}})+e^{i(\frac{\pi}{4}-\frac{\pi}{4})}}$$

$$= \frac{1-2e^{-i\frac{\pi}{4}}+e^{-i\frac{\pi}{2}}}{1-2\cos\frac{\pi}{4}+1} = \frac{1-2(\frac{\sqrt{2}}{2}-i\frac{\sqrt{2}}{2})+i}{2-\sqrt{2}} = \frac{i-1}{\sqrt{2}}.$$

□

Zadatak 0.12. *Dokazati da za* $x \neq 2k\pi$, $k \in Z$, *važi:*

$$\cos x + \cos 2x + ... + \cos nx = \frac{\cos \frac{n+1}{2} x \sin \frac{nx}{2}}{\sin \frac{x}{2}},$$

$$\sin x + \sin 2x + ... + \sin nx = \frac{\sin \frac{n+1}{2} x \sin \frac{nx}{2}}{\sin \frac{x}{2}}.$$

Rešenje. Neka je

$$\begin{aligned} S &= \cos x + \cos 2x + ... + \cos nx, \\ T &= \ \sin x + \sin 2x + ... + \sin nx. \end{aligned}$$

Tada je

$$\begin{aligned} S + iT &= (\cos x + i \sin x) + (\cos 2x + i \sin 2x) + \cdots + (\cos nx + i \sin nx) \\ &= e^{xi} + e^{2xi} + \cdots + e^{nxi} = e^{xi}\left(1 + e^{xi} + \cdots + e^{(n-1)xi}\right) = e^{xi}\frac{e^{nxi} - 1}{e^{xi} - 1} \\ &= e^{xi}\frac{2ie^{\frac{nxi}{2}} \sin \frac{nx}{2}}{2ie^{\frac{xi}{2}} \sin \frac{x}{2}} = e^{\frac{n+1}{2}xi} \cdot \frac{\sin \frac{nx}{2}}{\sin \frac{x}{2}} = \frac{\sin \frac{nx}{2}}{\sin \frac{x}{2}}\left(\cos \frac{n+1}{2}x + i \sin \frac{n+1}{2}x\right). \end{aligned}$$

Izjednačavajući realne i imaginarne delove u ova dva broja dobijamo tražene formule. □

Zadatak 0.13. *Izračunati sumu:*

$$1 + a\cos\varphi + a^2\cos 2\varphi + \cdots + a^n \cos n\varphi.$$

Rešenje. Neka je

$$\begin{aligned} S &= 1 + a\cos\varphi + a^2\cos 2\varphi + \cdots + a^n \cos n\varphi, \\ T &= \ a\sin\varphi + a^2\sin 2\varphi + \cdots + a^n \sin n\varphi. \end{aligned}$$

Tada za kompleksan broj $z = S + iT$ važi

$$\begin{aligned} z = 1 + (a\cos\varphi + ia\sin\varphi) + \left(a^2\cos 2\varphi + ia^2\sin 2\varphi\right) + \\ + \cdots + (a^n\cos n\varphi + ia^n \sin n\varphi) = \\ = 1 + ae^{\varphi i} + \left(ae^{\varphi i}\right)^2 + \cdots + \left(ae^{\varphi i}\right)^n = \frac{\left(ae^{\varphi i}\right)^{n+1} - 1}{ae^{\varphi i} - 1}. \end{aligned}$$

Kako je

$$\overline{ae^{\varphi i}-1}=\overline{a}\overline{e^{\varphi i}}-\overline{1}=a\overline{e^{\varphi i}}-1=ae^{-\varphi i}-1,$$

to je

$$z=\frac{\left(ae^{\varphi i}\right)^{n+1}-1}{ae^{\varphi i}-1}\cdot\frac{ae^{-\varphi i}-1}{ae^{-\varphi i}-1}=\frac{a^{n+2}e^{n\varphi i}-a^{n+1}e^{(n+1)\varphi i}-ae^{-\varphi i}+1}{a^2-a\left(e^{\varphi i}+e^{-\varphi i}\right)+1}.$$

Dalje, iz $e^{\varphi i}+e^{-\varphi i}=2\cos\varphi$ dobijamo

$$z=\frac{a^{n+2}e^{n\varphi i}-a^{n+1}e^{(n+1)\varphi i}-ae^{-\varphi i}+1}{a^2-2a\cos\varphi+1}.$$

Iz činjenice da je $S=\mathrm{Re}z$, dobijamo da je

$$S=\frac{a^{n+2}\cos n\varphi-a^{n+1}\cos\left(n+1\right)\varphi-a\cos\varphi+1}{a^2-2a\cos\varphi+1}.$$

□

Zadatak 0.14. *Neka je* $\omega=-\frac{1}{2}+i\frac{\sqrt{3}}{2}$.

a) *Dokazati da je* $\omega^3=1$.

b) *Dokazati da je* $\omega^2+\omega+1=0$.

c) *Izračunati* $\left(a+b\omega+c\omega^2\right)\left(a+b\omega^2+c\omega\right)$.

Rešenje. a) Odredimo ρ i φ za kompleksan broj $\omega=-\frac{1}{2}+i\frac{\sqrt{3}}{2}$.

$$\rho=\sqrt{x^2+y^2}=\sqrt{\left(-\tfrac{1}{2}\right)^2+\left(\tfrac{\sqrt{3}}{2}\right)^2}=1$$

$\varphi_0=arctg\left|\frac{y}{x}\right|=\varphi_0=arctg\left|\frac{\frac{\sqrt{3}}{2}}{\frac{1}{2}}\right|=\varphi_0=arctg\left|\sqrt{3}\right|=\frac{\pi}{3}\Longrightarrow\varphi=\pi-\frac{\pi}{3}=\frac{2\pi}{3}$

Dakle, u eksponencijalnom obliku kompleksan broj ω dat je sa $\omega=e^{i\frac{2\pi}{3}}$, pa je

$$\omega^3=\left(e^{i\frac{2\pi}{3}}\right)^3=e^{3\cdot i\frac{2\pi}{3}}=e^{2\pi i}=\cos 2\pi+i\sin 2\pi=1.$$

b) Na osnovu tvrđenja a), imamo da je $\omega^3 = 1$, odnosno $\omega^3 - 1 = 0$. Odavde dalje dobijamo da je $(\omega - 1)(\omega^2 + \omega + 1) = 0$, što je ekvivalentno sa $\omega = 1$ ili $\omega^2 + \omega + 1 = 0$. Kako je $\omega \neq 1$ (jer je $\omega = e^{i\frac{2\pi}{3}}$), to je $\omega^2 + \omega + 1 = 0$.

c) Na osnovu tvrđenja b), imamo da je $\omega^2 + \omega = -1$, pa je

$$\begin{aligned} M &= \left(a + b\omega + c\omega^2\right)\left(a + b\omega^2 + c\omega\right) \\ &= a^2 + ab\omega + ac\omega^2 + ab\omega^2 + b^2\omega^3 + bc\omega^4 + ac\omega + bc\omega + c^2\omega^3 \\ &= a^2 + b^2 + c^2 + (ab + ac + bc)\omega + (ab + ac + bc)\,\omega^2 \\ &= a^2 + b^2 + c^2 + (ab + ac + bc)\left(\omega + \omega^2\right) \\ &= a^2 + b^2 + c^2 - (ab + ac + bc). \end{aligned}$$

□

Zadatak 0.15. *Izvesti formule:*

$$\cos\frac{\pi}{10} = \frac{\sqrt{10 + 2\sqrt{5}}}{4} \qquad i \qquad \sin\frac{\pi}{10} = \frac{\sqrt{5} - 1}{4}.$$

Rešenje. Rešenja jednačine $z^5 - 1 = 0$ u polju $\mathbb{C}$ su $z_k = e^{i\frac{2k\pi}{5}}$, $k = 0, 1, 2, 3, 4$, odnosno

$$\begin{aligned} z_0 &= 1, \\ z_1 &= e^{\frac{2\pi}{5}i} = \cos\frac{2\pi}{5} + i\sin\frac{2\pi}{5}, \\ z_2 &= e^{\frac{4\pi}{5}i} = \cos\frac{4\pi}{5} + i\sin\frac{4\pi}{5} = -\cos\frac{\pi}{5} + i\sin\frac{\pi}{5}, \\ z_3 &= e^{\frac{6\pi}{5}i} = \cos\frac{6\pi}{5} + i\sin\frac{6\pi}{5} = -\cos\frac{\pi}{5} - i\sin\frac{\pi}{5}, \\ z_4 &= e^{\frac{8\pi}{5}i} = \cos\frac{8\pi}{5} + i\sin\frac{8\pi}{5} = \cos\frac{2\pi}{5} - i\sin\frac{2\pi}{5}. \end{aligned}$$

Primetimo da je

$$\begin{aligned} &\mathrm{Re}(z_1) > 0, \quad \mathrm{Im}(z_1) > 0; \\ &\mathrm{Re}(z_2) < 0, \quad \mathrm{Im}(z_2) > 0; \\ &\mathrm{Re}(z_3) < 0, \quad \mathrm{Im}(z_3) < 0; \\ &\mathrm{Re}(z_4) > 0, \quad \mathrm{Im}(z_4) < 0. \end{aligned}$$

Sada ćemo odrediti rešenja jednačine $z^5-1=0$ na drugi način i uporedićemo ih sa prethodno dobijenim rešenjima. Kako je $z^5-1=(z-1)(z^4+z^3+z^2+z+1)$, to je $z^5-1=0$ ekvivalentno sa $z-1=0$ ili $z^4+z^3+z^2+z+1=0$. Rešenje prve jednačine je $\tilde{z}_0=1$. Potražimo rešenja druge jednačine.

$$\begin{aligned} z^4+z^3+z^2+z+1&=0\\ z^4+z^3+z^2+z+1&=0 \quad /:z^2\\ z^2+z+1+\frac{1}{z}+\frac{1}{z^2}&=0\\ \left(z^2+\frac{1}{z^2}\right)+\left(z+\frac{1}{z}\right)+1&=0 \end{aligned}$$

Ako uvedimo smenu $z+\frac{1}{z}=t$, pri kojoj je $z^2+\frac{1}{z^2}=t^2-2$, prethodna jednačina se svodi na kvadratnu jednačinu $t^2+t-1=0$, čija su rešenja $t_{1,2}=\frac{-1\pm\sqrt{5}}{2}$. U prvom slučaju, ako je $z+\frac{1}{z}=-\frac{\sqrt{5}+1}{2}$, odnosno $2z^2+\left(\sqrt{5}+1\right)z+2=0$, dobijaju se dva rešenja

$$\tilde{z}_{1,2}=\frac{-(\sqrt{5}+1)\pm\sqrt{(\sqrt{5}+1)^2-16}}{4}=-\frac{\sqrt{5}+1}{4}\pm i\frac{\sqrt{10-2\sqrt{5}}}{4},$$

a u drugom slučaju, ako je $z+\frac{1}{z}=\frac{\sqrt{5}-1}{2}$, odnosno $2z^2-(\sqrt{5}-1)z+2=0$, dobijaju se rešenja

$$\tilde{z}_{3,4}=\frac{\sqrt{5}-1\pm\sqrt{(\sqrt{5}-1)^2-16}}{4}=\frac{\sqrt{5}-1}{4}\pm i\frac{\sqrt{10+2\sqrt{5}}}{4}.$$

Primetimo da je

$$\begin{aligned} &\mathrm{Re}(\tilde{z}_1)<0, \quad \mathrm{Im}(\tilde{z}_1)<0;\\ &\mathrm{Re}(\tilde{z}_2)<0, \quad \mathrm{Im}(\tilde{z}_2)>0;\\ &\mathrm{Re}(\tilde{z}_3)>0, \quad \mathrm{Im}(\tilde{z}_3)>0;\\ &\mathrm{Re}(\tilde{z}_4)>0, \quad \mathrm{Im}(\tilde{z}_4)<0, \end{aligned}$$

pa mora biti $\tilde{z}_1=z_3$, $\tilde{z}_2=z_2$, $\tilde{z}_3=z_1$ i $\tilde{z}_4=z_4$. Odavde dalje imamo da je

$$\begin{aligned} \cos\frac{\pi}{5}&=\frac{\sqrt{5}+1}{4}, & \sin\frac{\pi}{5}&=\frac{\sqrt{10-2\sqrt{5}}}{4},\\ \cos\frac{2\pi}{5}&=\frac{\sqrt{5}-1}{4}=\sin\frac{\pi}{10}, & \sin\frac{2\pi}{5}&=\frac{\sqrt{10+2\sqrt{5}}}{4}=\cos\frac{\pi}{10}. \end{aligned}$$

□

Zadatak 0.16. *Ako je* $|z_1| = |z_2| = |z_3|$, *tada je* $\arg \dfrac{z_3 - z_2}{z_3 - z_1} = \dfrac{1}{2} \arg \dfrac{z_2}{z_1}$. *Dokazati.*

Rešenje. Neka je $z_j = \rho e^{i\varphi_j}$, za $j = 1, 2, 3$. Tada je

$$\frac{z_3 - z_2}{z_3 - z_1} = \frac{\rho e^{i\varphi_3} - \rho e^{i\varphi_2}}{\rho e^{i\varphi_3} - \rho e^{i\varphi_1}} = \frac{e^{i\varphi_3} - e^{i\varphi_2}}{e^{i\varphi_3} - e^{i\varphi_1}} = \frac{-e^{i\varphi_3}\left(e^{i(\varphi_2 - \varphi_3)} - 1\right)}{-e^{i\varphi_3}\left(e^{i(\varphi_1 - \varphi_3)} - 1\right)}$$

$$= \frac{2ie^{i\frac{\varphi_2 - \varphi_3}{2}} \sin \frac{\varphi_2 - \varphi_3}{2}}{2ie^{i\frac{\varphi_1 - \varphi_3}{2}} \sin \frac{\varphi_1 - \varphi_3}{2}} = \frac{\sin \frac{\varphi_2 - \varphi_3}{2}}{\sin \frac{\varphi_1 - \varphi_3}{2}} \cdot e^{i\frac{\varphi_2 - \varphi_1}{2}},$$

pa sledi da je $\arg \dfrac{z_3 - z_2}{z_3 - z_1} = \dfrac{\varphi_2 - \varphi_1}{2}$.

Sa druge strane je

$$\frac{1}{2} \arg \frac{z_2}{z_1} = \frac{1}{2} \arg \frac{\rho e^{i\varphi_2}}{\rho e^{i\varphi_1}} = \frac{1}{2} \arg e^{i(\varphi_2 - \varphi_1)} = \frac{1}{2} \left(\varphi_2 - \varphi_1\right),$$

pa važi

$$\arg \frac{z_3 - z_2}{z_3 - z_1} = \frac{1}{2} \arg \frac{z_2}{z_1}.$$

□